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LOCALIZATION IN NOETHERIAN P.I. RINGS AS A FINITE CRITERION

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ABSTRACT

Several equivalent (finite) conditions for the localizability of a prime ideal in a noetherian prime p.i. ring are given. It is used to connect localizability to the curve criterion. Another application is to some integrality questions.

1. Introduction and notations

One of the main results in [BW] is the surprising fact that the determination of the localizability of a prime (semi-prime) ideal in a prime noetherian p.i. ring is a finite procedure. Explicitly, let $P \in \operatorname{spec} R$ be a prime ideal and let $Q_1, \ldots, Q_r$ be all the prime ideals in $T(R)$ contracting to P. We say that:

$$\{P\} \text{ is tr-closed, if } Q \in \operatorname{spec} T(R),\ Q \cap Z(T(R))$$
$$= Q_j \cap Z(T(R)) \text{ for some } j \Rightarrow Q = Q_k \text{ for some } k.$$

Then the above-mentioned result is the equivalence of the following statements:

(1) P is right (left) localizable,
(2) $\{P\}$ is tr-closed.

The purpose of the present paper is to give some more equivalent conditions to (1) (and (2)), using a more direct elementwise approach. In doing so we are able as well to put localizability in the context of integrality and relate it to the curve criteria. In order to give a sample of our results we need some more conventions as follows.

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Let V be a semi-prime ideal and let

$$\tau(V) \equiv \{x \in R \mid \bar{x} \text{ is regular in } R/V\}.$$

For any $x \in R$ let $x^n - c_1(x)x^{n-1} + \cdots \pm c_n(x) = 0$ be the Cayley–Hamilton equation of x where $n = \text{p.i.d.}(R) \equiv$ the p.i. degree of R, of course $c_1(x) = \text{tr}(x)$, $c_n(x) = \det(x)$. Define

$$T(R) = R[c_i(x) \mid x \in R, 1 \leqq i \leqq n] \qquad \text{(e.g. [R]).}$$

Let $P \in \operatorname{spec} R$ and $P'_1, \ldots, P'_k$ are all the prime ideals in $T(R)$ contracting to P. Let $S = \tau(P)$. As a consequence of our main result we obtain the following

THEOREM A. *Let R be a prime noetherian p.i. ring, $P \in \operatorname{spec} R$ and S, $P'_1, \ldots, P'_k$ as above. Then the following are equivalent:*

(i) $\{P\}$ *is* tr*-closed,*

(ii) $\det(S) \cap \bigcup_{i=1}^k P'_i = \varnothing$ $(\det(S) \equiv \{\det(t) \mid t \in S\})$,

(iii) $c_i(x) \in \bigcap_{i=1}^k P'_i$, *for all* $x \in P$, $1 \leqq i \leqq n$,

(iv) *P is left (and right) localizable.*

Recall that Schelter ([Sch]) has shown that for a noetherian prime p.i. ring R one has for all $x \in R$, $i \leqq i \leqq n$,

$$(c_i(x))^m + (c_i(x))^{m-1}r_1 + \cdots + c_i(x)r_{m-1} + r_m = 0$$

where $r_i \in R$ (of course m and $r_1, \ldots, r_m$ depend on $c_i(x)$). This is the integrality of $T(R)$ over R. In this context we have the following:

THEOREM B. *Let R be a prime noetherian p.i. ring and M a maximal ideal in R. Then the following are equivalent:*

(1) *M is right (left) localizable,*

(2) *For every $x \in M$, $1 \leqq i \leqq n$ there exist $p_1, \ldots, p_m \in M$ such that*

$$(c_i(x))^m + (c_i(x))^{m-1}p_1 + \cdots + p_m = 0.$$

Another consequence is our next result, relating localizability to the curve criteria (e.g., [ASc]).

THEOREM C. *Let R be a prime noetherian affine p.i. ring,* p.i.d. $(R) = n$, *M a maximal ideal in R. Then the following are equivalent:*

(1) *M is right (left) localizable,*

(2) *For any prime P, $P \subset M$ satisfying*

 (i) $\text{k.dim}(R/P) = 1$,

 (ii) p.i. degree $(R/P) = n$.

Then M/P is localizable in R/P.

In order to emphasize the finiteness of our criterion we mention the next result, which is an immediate corollary of our main theorm, Sirsov's theorem and Newton's formulae.

COROLLARY. *Let $R = F\{x_1, \dots, x_k\}$ be a prime noetherian p.i. ring,* p.i. degree $(R) = n$, *and $P = \Sigma_{i=1}^{N} g_i R$, a prime ideal in R. Suppose that* char $F = 0$. *Let $P'_1, \dots, P'_s$ be all the prime ideals in $T(R)$ contracting to P. Then the following are equivalent*:

(1) *P is left (right) localizable,*

(2) $\mathrm{tr}(g_i M(x_1, \dots, x_k)) \in \bigcap_{j=1}^{s} P'_j$, $1 \leqq i \leqq N$ *for all the monomials $M(x_1, \dots, x_k)$ (finite in number) in $x_1, \dots, x_k$ of degree $\leqq \beta$, where β is determined by Shirsov's Theorem* (e.g., [R, p. 205]).

2. The proof of the main results

Let $V = P_1 \cap \cdots \cap P_r$ be a semiprime ideal (irredundant intersection) and let $P'_1, \dots, P'_k$ be all the prime ideals in $T(R)$ contracting to $P_1, \dots, P_r$. Also, let $S \equiv \tau(V)$. We shall now state our main result in its full generality:

MAIN THEOREM. *Let R be a prime noetherian p.i. ring $V = P_1 \cap \cdots \cap P_r$, a semiprime ideal such that* $\mathrm{k.dim}(R/P_1) = \cdots = \mathrm{k.dim}(R/P_r)$ (*where* k.dim( ) *denotes the classical Krull dimension*). *Let S and $P'_1, \dots, P'_k$ be as above. Then the following are equivalent*:

(i) $\{P_1, \dots, P_r\}$ *is* tr-*closed,*

(ii) $\det(S) \cap \bigcup_{i=1}^{k} P'_i = \varnothing$,

(iii) $c_i(x) \in \bigcap_{i=1}^{k} P'_i$, *for all $x \in V$, $1 \leqq i \leqq n$,*

(iv) *V is left (right) localizable.*

More generally the following implications hold (without assuming $\mathrm{k.dim}(R/P_i) = \mathrm{k.d.}(R/P_j)\ \forall i, j$):

$$
\begin{array}{c}
\text{(iii)} \Leftarrow \text{(i)} \Leftrightarrow \text{(ii)} \\
\Updownarrow \\
\text{(iv)}
\end{array}
$$

THE PROOF OF THE MAIN THEOREM. The proof consists of separate steps, establishing each of the implications.

(i)$\Leftrightarrow$(iv): This is immediate in view of [BW]. Q.E.D.

(i)$\Rightarrow$(iii): Let $p_i = P'_i \cap Z(T(R))$ and assume by reindexing that $P'_1, \ldots, P'_t$, $t \leqq k$, are all the primes among $\{P'_1, \ldots, P'_k\}$ contracting to p_1. By (i)

$$\mathrm{Jac}(p_{1_{p_1}} T(R)_{p_1}) = P'_{1_{p_1}} \cap \cdots \cap P'_{t_{p_1}}.$$

Let $x \in V$, so $x \in \bigcap_{i=1}^{t} P'_{i_{p_1}}$. By ([B, Lemma 1, Cor 7]), $c_i(x) \in p_{1_{p_1}}$, for $1 \leqq i \leqq n$.

(In fact Lemma 1 in [B] establishes the required result for $R = \Lambda\{x_1, \ldots, x_s\}$; using Lemma 1, with the aid of Corollary 7, one establishes the general case.)

Now $p_{1_{p_1}} \cap Z(T(R)) = p_1$ implies that $c_i(x) \in p_1$, $1 \leqq i \leqq n$. An identical argument will give $c_i(x) \in \bigcap_{j=1}^{t} p_i$, $1 \leqq i \leqq n$. Q.E.D.

(ii)$\Rightarrow$(i): Suppose we ordered $P_1, \ldots, P_r$ in such a way that

$$\begin{aligned} \text{k.dim}\, R/P_1 = \cdots = \text{k.dim}\, R/P_{i_1} &< \text{k.dim}\, R/P_{i_1+1} \\ = \cdots = \text{k.dim}\, R/P_{i_2} &< \text{k.dim}\, R/P_{i_2+1} \cdots \end{aligned}$$

that is, into equi-Krull dimension subsets. We shall prove that each such equi-Krull dimension subset satisfies (i) and consequently so will their union $\{P_1, \ldots, P_r\}$. Say $\{P_{a+1}, \ldots, P_b\}$ is our given subset ($a = i_x$, $b = i_{x+1}$ for some x) and the validity of (i) was proved for all the previous subsets. Let $Q'_1, \ldots, Q'_m$ be all the primes in $T(R)$ contracting to $\{P_{a+1}, \ldots, P_b\}$ (a subset of $P'_1, \ldots, P'_k$). Let $Q \in \mathrm{spect}\; T(R)$ satisfy $Q \cap Z(T(R)) = Q'_j \cap Z(T(R))$ for some j. We need to show that $Q \cap R \in \{P_{a+1}, \ldots, P_b\}$. By choice $Q'_j \cap R = P_y$ for some $a + 1 \leqq y \leqq b$.

Suppose that $Q \cap R \not\subset \bigcup_{i=1}^{r} P_i$. So $Q \cap R$ projects in $R/\bigcap_{i=1}^{r} P_i$, to an ideal which contains a regular element. Hence there exists $t \in S \equiv \tau(V)$ such that $t \in Q \cap R$. By the Cayley–Hamilton equation, we get that $\det(t) \in Q$ and consequently

$$\det(t) \in Q \cap Z(T(R)) = Q'_j \cap Z(T(R))$$

violating (ii). Thus $Q \cap R \subset \bigcup_{i=1}^{r} P_i$ and therefore $Q \cap R \subset P_c$ for some c. We need to treat two cases separately.

Case 1. $Q \cap R = P_c$. Hence

$$\text{k.dim}\left(\frac{R}{Q\cap R}\right)=\text{k.dim}\left(\frac{T(R)}{Q}\right)=\text{k.dim}\ \frac{Z(T(R))}{Q\cap Z(T(R))}$$

$$=\text{k.dim}\left(\frac{T(R)}{Q_j'}\right)=\text{k.dim}\frac{R}{P_y}$$

implying that k.dim $R/P_c=$ k.dim R/P_y and so $Q\cap R=P_c\in\{P_{a+1},\dots,P_b\}$.

Case 2. Suppose $Q\cap R\subset P_c$. We have

$$\text{k.dim}\ \frac{R}{Q_j'\cap R}=\text{k.dim}\ \frac{T(R)}{Q_j'}=\text{k.dim}\ \frac{T(R)}{Q}=\text{k.dim}\ \frac{R}{Q\cap R}\gneqq\text{k.dim}\ \frac{R}{P_c}$$

and therefore P_c belongs to one of the previous equi-Krull dimension subsets. By G.U. between R and $T(R)$, there exists $W\in\operatorname{spec}T(R)$ satisfying $W\cap R=P_c$ and $W\supsetneqq Q$. Hence

$$w\equiv W\cap Z(T(R))\supsetneqq q\equiv Q\cap Z(T(R)).$$

By G.U. between $Z(T(R))$ and $T(R)$ there exists $W_1\in\operatorname{spec}T(R)$ satisfying $W_1\cap Z(T(R))=w$, $W_1\supset Q_j'$ $(q=Q\cap Z(T(R))=Q_j'\cap Z(T(R)))$. But (i) holds for the previous equi-Krull dimension sets and $W\cap R=P_c$ implies that $W_1\cap R\equiv P_d$ belongs to the same subset to which P_c belongs since

$$\text{k.dim}\ \frac{R}{P_c}=\text{k.dim}\ \frac{T(R)}{W}=\text{k.dim}\ \frac{Z(T(R))}{w}=\text{k.dim}\ \frac{T(R)}{W_1}=\text{k.dim}\ \frac{R}{W_1\cap R}.$$

Thus $P_d=W_1\cap R\supsetneqq Q_j'\cap R=P_y$, violating the irredundancy of $\{P_1,\dots,P_r\}$.

Q.E.D.

(iii)⇒(ii): We shall prove it using the assumption:

$$\text{k.dim}\ R/P_1=\cdots=\text{k.dim}(R/P_r).$$

In fact this is the only place where this assumption is being used. In order to do so we need the following consequence of a result of [St].

Proposition. *Let R be a prime p.i. noetherian ring and X a finite union of cliques* (e.g., [J], [Br]) *of prime ideals in R. Then $X[t]\equiv\{P[t]\mid P\in X\}$ is localizable in $R[t]$, the polynomial ring with variable t over R.*

Sketch of Proof. There are two major steps in the proof. The first one, due to Stafford and Goodearl, is the following

LEMMA. *Let R be a noetherian (p.i.) ring. Let P_0 be a prime ideal in R and Q a prime ideal in $R[t]$ such that $P_0[t] \rightsquigarrow Q$ ("linked"). Then there is a prime ideal Q_0 of R such that $Q = Q_0[t]$ and either $P_0 = Q_0$ or $P_0 \rightsquigarrow Q_0$.*

We shall not give an indication how this Lemma is actually proved, since it uses methods quite different from the one used in the present paper. We shall merely say that as a consequence, $X[t]$ is a finite union of cliques in $R[t]$.

The second step, due to Stafford, is to show that $X[t]$ satisfies the intersection condition, that is, if J is an ideal in $R[t]$ and $J \cap \tau(P[t]) \neq \varnothing$ for all $P \in X$, then $J \cap (\cap \{\tau(P[t]) \mid P \in X\}) \neq \varnothing$.

Again we shall not give a proof of the result in its full generality. Instead, we shall give an alternative argument, valid for a commutative R, an argument which can be successfully generalized to the more general p.i. case. Assume further (for the sake of simplification) that $J = I[t]$. Let $I = a_1R + \cdots + a_dR$. Let

$$s = a_1t^{i_1} + \cdots + a_dt^{i_d},$$

where $i_1 < i_2 < \cdots < i_d$. Now, by assumption $I \not\subset P$, for any $P \in X$. Consequently, $s \notin P[t]$ and therefore $s \in J \cap (\cap \{\tau(P[t] \mid P \in X\})$.

Finally, by Jategaonkar's theorem [e.g. J, 8.36] one obtains that $X[t]$ is localizable. Q.E.D.

REMARK. If one is willing to assume, in addition, that R contains an uncountable field or that R is finitely generated over a central noetherian subring, then the previous proposition is superfluous. In fact, by [M] and [Br, Theorem 3.7], X is already localizable.

It is clear that if $S_1 = \tau(P_1[t] \cap \cdots \cap P_r[t])$, then $\det(S_1) \cap \bigcup_{i=1}^k P_i'[t] = \varnothing$ will imply that $\det(S) \cap \bigcup_{i=1}^k P_i' = \varnothing$. Moreover, by (iii) (using, e.g., [A]), one can easily verify that

$$c_i(y) \in \bigcap_{i=1}^{k} P_i'[t] \qquad \text{for all } y \in \bigcap_{i=1}^{r} P_i[t], \quad 1 \leqq i \leqq n.$$

Finally $T(R[t]) = T(R)[t]$ and so $P_1'[t], \ldots, P_k'[t]$ are all the prime ideals in $T(R[t])$ contracting to $P_1[t], \ldots, P_r[t]$.

Let $X(P_i)$ = the clique of P_i and $X \equiv X(P_1) \cup \cdots \cup X(P_r)$. Then the previous reasoning shows that we may assume that X is localizable, that is $S_0 = \tau(X) \equiv \cap \{\tau(Q) \mid Q \in X\}$ is an Ore set. Now R is prime and noetherian, and $\tau(X)$ is also an Ore set in $T(R)$, implying by [BS] that $T(R)_{\det(S_0)} = T(R_{S_0}) = T(R)_{S_0}$, is integral over R_{S_0}. Consequently, if $Q \in X$, Q survives in R_{S_0}, hence in $T(R_{S_0}) = T(R)_{\det(S_0)}$ and therefore $\det(S_0) \cap Q = \emptyset$.

Suppose, by negation, that there exists $t \in S$ and $\det(t) \in P'_j$ for some $1 \leqq j \leqq k$. It is easily checked that $V_{S_0} = \bigcap_{i=1}^{r} P_{i_{S_0}}$. Now since $\mathrm{k.dim}(R/P_1) = \cdots = \mathrm{k.dim}(R/P_r)$, we have that $\{P_{i_{S_0}}\}$ are all maximals in R_{S_0} and consequently

$$R_{S_0}/V_{S_0} \cong R_{S_0}/P_{1_{S_0}} \oplus \cdots \oplus R_{S_0}/P_{r_{S_0}}.$$

Also $R/V \subset R_{S_0}/V_{S_0}$ and obviously quotient ring $(R/V) = R_{S_0}/V_{S_0}$. Clearly, $\bar{t}$, the image of t in R/V, is a non-zero divisor and therefore there exists $z \in S_0$ and $u \in R$ satisfying $\bar{t}\bar{u} = \bar{z}$. Equivalently $tu - z \equiv x \in V$, or $tu = z + x$. Now, by [A] we have

$$\det(z + x) = \det(z) + \det(x) + \sum c_{j_1}(\beta_1) \cdots c_{j_e}(\beta_e)$$

where β_i's are monomials in z and x, with x appearing nontrivially. Consequently, $\beta_i \in V$ for each i and by (iii) we have that

$$\det(z + x) = \det(z) + \alpha, \qquad \alpha \in \bigcap_i P'_i.$$

Now $\det(z) + \alpha = \det(z + x) = \det(tu) = \det(t)\det(u) \in P'_j$, since $\det(t) \in P'_j$. Consequently $\det(z) \in P'_j$. But $S_0 \cap (P'_j \cap R) = \emptyset$, implies that P'_j survive in $T(R)_{S_0} = T(R)_{\det(S_0)}$, a contradiction to $\det(z) \in P'_j$. Q.E.D.

Finally, we need to show that (i)$\Rightarrow$(ii). Equivalently, we shall show that (iv) implies (ii). We clearly have that R_S is noetherian, S is Ore in $T(R)$ and $T(R)_{\det(S)} = T(R_S) = T(R)_S$ is integral over R_S, where the equalities are due to [BS, Lemmas 1, 2]. So, given P'_i, $1 \leqq i \leqq k$, $(P'_i \cap R) \cap S = \emptyset$ and consequently $P'_{i_S} \neq T(R)_S$, but by the above-mentioned equalities $P'_{i_S} = P'_{i_{\det(S)}}$, implying that $\det(S) \cap P'_i = \emptyset$. Q.E.D.

REMARK. *On the validity of* (iii)$\Rightarrow$(i) (*or* (ii)) *in general*

The implication (iii)$\Rightarrow$(i) does not hold in general. Indeed, let R be a prime noetherian p.i. ring. It is not difficult to show, using [B], that Jac(R), its Jacobson's radical, satisfies (iii). However, there exists an example, due to Warfield (unpublished as yet), too comlicated to be described here, of a prime noetherian p.i. ring R with a nonlocalizable Jac(R).

PROOF OF THEOREM B. We shall prove at first the implication (2)$\Rightarrow$(1). Say $P'_1, \ldots, P'_k$ are all the prime (= maximals) in $T(R)$ contracting to M. By

(2) $c_i^m(x) \in MT(R)$ $\forall x \in M$, $1 \leqq i \leqq n$. Consequently $c_i(x) \in \bigcap_{i=1}^{k} P_i'$ for all $x \in M$, $1 \leqq i \leqq n$, so by (iii) of the main theorem we get that M is left and right localizable.

To prove the implication $(1) \Rightarrow (2)$, we use again the established fact in the main theorem, that localizability is equivalent to (iii). Hence, since $\mathrm{rad}(MT(R)) = P_1' \cap \cdots \cap P_k'$, for every $x \in M$, $1 \leqq i \leqq n$ there exists an integer m_1 such that $c_i(x)^{m_1} \in MT(R)$. The required conclusion will now follow, provided we prove the following.

Lemma. *Let R be a prime noetherian p.i. ring, I an ideal in R. Then every element of $IT(R)$ is integral (a la Schelter* [Sch]) *over R with "coefficients" in I.*

The following result is due to Pare and Schelter:

Proposition [PS]. *Let $S = Ru_1 + \cdots + Ru_k$ be a centralizing extension. Then each element in S is integral over R.*

We shall actually prove a more general statement:

Proposition. *Let R be a noetherian p.i. ring and S a finite central extension. Let I be an ideal in R. Then every element in IS is integral over R with "coefficients" in I.*

Proof. By induction on the lexicographicly ordered pair $\langle \text{p.i.d}(R/N(R)), d(R) \rangle$ where $d(R)$ is defined as follows:

Let a be the set of all evaluations of Formanek's central polynomial in $R/N(R) \equiv \bar{R}$ evaluated on $I + N(R)/N(R)$. Then $a\bar{R} = g_1\bar{R} + \cdots + g_d\bar{R}$, $g_i \in a$, $i = 1, \ldots, d$ (we pick d to be minimal such).

By an easy reduction we may assume that R, S are prime. If $\langle \text{p.i.d.}(R), d \rangle = \langle 1, d \rangle$, then R and S are commutative and the result follows from the usual commutative arguments. Clearly we have that $Z(R) \subset Z(S)$ and so $a \subseteq Z(S)$. Let $K \equiv (g_1R)S = g_1S$. Then $R/K \cap R \subset S/K$ is a finite central extension.

Claim. *$IS + K/K$ is integral with "coefficients" in I over $R/K \cap R$.*

Proof. Let $Q_1, \ldots, Q_r$ be the prime ideals in S containing K. Enough to show this in IS/Q_j over $R/Q_j \cap R$. If p.i.d. $R/Q_j \cap R \lneqq$ p.i.d.(R), then the result follows by induction.

Suppose that p.i.d.$(R/Q_j \cap R) =$ p.i.d.(R). Then $\bar{g}_2\bar{R} + \cdots + \bar{g}_d\bar{R} =$ the ideal in $R/Q_j \cap R \equiv \bar{R}$ generated by all the evaluations of Formanek's central polynomial on $I + Q_j \cap R/Q_j \cap R$. Consequently

$$\langle \text{p.i.d.}(\bar{R}), d(\bar{R})\rangle \leqq \langle \text{p.i.d.}(\bar{R}), d(R)-1\rangle \lneqq \langle \text{p.i.d.}(R), d(R)\rangle$$

and we get the result by induction. Q.E.D.

The theorem will now follow if we can show that each element in $K = g_1S$ is integral (with coefficients in I) over R. Let $x \in S$. Then, by the previous proposition, there exists $m_1, \dots, m_t(x)$ such that $x^t + m_1(x) + \cdots + m_t(x) = 0$, where $m_i(x) =$ sum of monomials of the form $r_{i1}xr_{i2}x \cdots xr_{i,t-i+1}$, $1 \leqq i \leqq t$. Consequently, if $y = g_1x \in K$ arbitrary, then multiply the above equality by g_1^t and get

$$(g_1x)^t + g_1m_1(g_1x) + \cdots + m_t(g_1x)g_1^t = 0 \qquad \text{or}$$

$$y^t + g_1m_1(y) + \cdots + m_t(y)g_1^t = 0 \qquad \text{Q.E.D.}$$

Remark. Theorem B remains true if we replace M by a finite intersection of maximal ideals.

In order to prove Theorem C we need the following

Lemma. *Let R be a noetherian, prime p.i. ring. Let P, Q be prime ideals in R, satisfying*

(1) $P \subset Q$,

(2) height$(Q/P) \gneqq \geqq 1$.

Then there are infinitely many primes $\{P_\alpha\}$ in R such that $P \subset P_\alpha \subsetneqq Q$, and height$(P_\alpha/P) = 1$, *for all* α.

Proof. This is a straightforward generalization of the same theorem for commutative rings, using the principal ideal theorem [R, p. 229].

Corollary. Let R be a noetherian prime p.i. ring, Q a prime ideal in R. Then, there exists a prime ideal P in R, satisfying

(1) p.i. degree(R/P) = p.i. degree(R),

(2) $P \subset Q$,

(3) height$(Q/P) = 1$.

Proof. This is done by induction on height(Q) and observing that if $\{P_\alpha\}$ is an infinite family of height 1 prime ideals, then all but finitely many satisfy (1).

We shall now prove Theorem C. Clearly, if M is localizable then M/P is localizable for any P prime, $P \subset M$. To prove the opposite direction, let $P_1', \dots, P_k'$ be all the prime (= maximals) in R contracting to M. By our main result we need to show that

$$c_i(x) \in \bigcap_{j=1}^{k} P_i', \qquad 1 \leqq i \leqq n \quad \text{for every } x \in M.$$

Let $p_i \equiv P_i' \cap Z(T(R))$, $i = 1, \ldots, k$. So, we need to prove that $c_i(x) \in \bigcap_{j=1}^{k} p_j$, $1 \leqq i \leqq n$. By the previous corollary there exists P', a prime ideal in $T(R)$, satisfying p.i. degree$(T(R)/P') = n$, $P' \subset P_1'$ and height$(P_1'/P') = 1$. Now, since R (and consequently $T(R)$) is affine we have that k.dim $T(R)/P' =$ height$(P', P') = 1$. Since R is noetherian and affine, we easily deduce that k.dim$(R/(P' \cap R)) = 1$. Now it is standard (e.g., [ASm]) that $T(R/P' \cap R) = T(R)/P'$. Also $P' \cap R \subsetneqq P_1' \cap R = M$. Consequently, assuming (2) of Theorem C we get that $c_i(\bar{x}) \in P_1'/P'$ for all $\bar{x} \in M/P' \cap R'$, $1 \leqq i \leqq n$. Now, since $c_i(\bar{x}) = \overline{c_i(x)}$, the image of $c_i(x)$ in $T(R)/P'$ (e.g., [ASm]) we conclude that $c_i(x) \in P_1'$, $1 \leqq i \leqq n$ for every $x \in M$. An identical proof yields the same for $P_2', \ldots, P_k'$, and consequently establishing the result. Q.E.D.

Remark. Theorem C is not true for a general prime noetherian p.i. ring. In fact, there are counter examples which are finite modules over their noetherian center. Indeed, let

$$R = \left\{ \begin{pmatrix} r + a, & b \\ c, & r \end{pmatrix} : r \in A, a, b \in M_1, c \in M \right\},$$

the major example in [S], where A is a commutative local ring with maximal ideal M, B is a finite A module with exactly two maximal ideals M_1, M_2 and $M_1 \cap M_2 = M$. Also, height$(M_1) = 1$, height$(M_2) = 2$.

Let

$$P = \begin{pmatrix} M_1 & M_1 \\ M & M \end{pmatrix}.$$

Then P is not a localizable maximal ideal, height$(P) = 2$. Let $V \subset P$ satisfying p.i. degree$(R/V) =$ p.i.d.(R) and k.d.$(R/V) = 1(=$ height$(P/V))$. Then, as is easily checked, P/V is localizable, in fact R/V is a local ring with a unique maximal ideal P/V.

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